A NEW PROOF OF THE WEIL IDENTITY

BEN ANTHES

Introduction

Among the known proofs of the celebrated Kähler identities, there is essentially one—due to André Weil [Wei58]—which it is almost completely linear algebraic. The earlier approaches were mostly calculations in coordinates. Weil reduces the problem to the so called Weil identity, which describes the interplay of the Hodge dual and the Lefschetz operators. There are some quite different proofs of this identity and we are going to give a new, more enlightening one. The original proof, given by Weil, is based on a technical lemma which controls the behaviour of forms of special type under the Hodge dual. Another quite technical, but tricky proof of the Weil identity can be found in [Huy05] and we are going to pick up the idea of using the fact that the exterior algebra of an orthogonal sum splits into the tensor product of the summands, which allows us to argue inductively. However, the calculations in the proof given by Huybrechts are still cumbersome and opaque. The first one giving a proof using the powerful tool of representation theory is Henryk Hecht; this proof is presented in [Wel80]. It is shown that the Hodge dual is related to the Weil reflection on the natural $\mathfrak{sl}_2$ representation of the exterior algebra; however, this approach requires differential methods and actually involves too much representation theory to clarify the Weil identity.

We will show that the Hodge dual essentially is another well understood operator of the theory of $\mathfrak{sl}_2$ representations: the *τ-operator (or Chevalley operator)*, which reflects the weighted spaces of those representations. This approach is due to the idea of Prof. Dr. Manfred Lehn, who kindly shared his thought with me.

§1. The Weil identity

To state the Identity, we need a few notations and operators on the exterior algebras $\bigwedge V^*$ and $\bigwedge V^*_{\mathbb{C}}$, where we think of V as the tangent space of a point on a Hermitian manifold, i.e., V is a n-dimensional Hermitian vector space and $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is the dual of the underlying real vector space, whilst $V^*_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C}$. The canonical complex structure on V has a dual complex structure, which canonically extends to the *Weil operator* $\mathrm{I}\colon \bigwedge V^*_{\mathbb{C}} \to \bigwedge V^*_{\mathbb{C}}$, which acts on (p, q)-forms by multiplication with i^{p-q} and is extended by linearity. In [Wei58], the Hodge dual is defined as linear operator on $\bigwedge V^*$. We will stick to this convention to arrive at exactly the same formula.

The canonical complex structure and the Hermitian inner product on V determine the associated fundamental 2-form $\omega \in \bigwedge^{1,1} V^* \cap \bigwedge^2 V^*$, and the Lefschetz operators on $\bigwedge V^*$ are given by $L\colon \alpha \mapsto \alpha \wedge \omega$ and $\Lambda = *^{-1} L *$, where the Hodge dual $*$ is defined with respect to the orientation $\mathrm{vol} = \omega^n/n!$ of norm 1.

The Weil identity now describes the following interaction of L and $*$ on homogenous forms annihilated by Λ, the so called *primitive* forms.

Weil identity (1.1) ([Wei58], Théorème I.2). *On the exterior algebra $\bigwedge V^*$ of a n-dimensional Hermitian vector space V, we have*

$$*L^j\alpha = \frac{(-1)^{\binom{n+1}{2}} j!}{(n-k-j)!} L^{n-k-j}\,\mathrm{I}\,\alpha$$

for all primitive k-forms $\alpha \in \bigwedge^k V^*$ (i.e., with $\Lambda\alpha = 0$,) and all $j \geqslant 0$.

Remark (1.2). This formula is mysterious for at least two reasons. Of course, it is not too surprising that there is an expression of $*L$, but we would expect a formula which holds for all (at least homogenous) forms and not just the primitive ones, and there still is this mysterious sign. We will derive a simple identity which holds for all forms and is essentially equivalent to the Weil identities when restricted to primitive forms, thus explaining this restriction to some extent.

One important observations for the proof is that for grading reasons, instead of proving the Weil identity for each j separately, we can equivalently show

$$*\exp(L)\alpha = (-1)^{\binom{n+1}{2}} \exp(L)\,\mathrm{I}\,\alpha$$

for any primitive α. The second key observation is that the Hodge dual has a certain representation-theoretic counterpart: Every $\mathfrak{sl}_2$ representation possesses a canonical isomorphism, known as the *τ- or Chevalley operator*, which plays an important role in the study of representations of semi-simple lie algebras. In our case, the operator is given by $\tau = \exp(L)\exp(-\Lambda)\exp(L)$. The following behaviour of the Chevalley operator is very important for us since it allows an inductive argument. Consider the case of V splitting into an orthogonal sum $V = W_1 \oplus W_2$. Then, on $\bigwedge V^* = \bigwedge W_1^* \otimes \bigwedge W_2^*$, the fundamental 2-form splits into the sum $\omega = \omega_1 + \omega_2$ of those on $\bigwedge W_1^*$ and $\bigwedge W_2^*$ respectively; thus, the Lefschetz operator $L = L_1 + L_2$ acts on the tensor product as $L_1 \otimes \mathrm{id} + \mathrm{id} \otimes L_2$, and since $\Lambda = \Lambda_1 + \Lambda_2$ is its adjoint, it analogously acts by $\Lambda_1 \otimes \mathrm{id} + \mathrm{id} \otimes \Lambda_2$. Hence, the natural representation on $\bigwedge V^*$ coincides with the product representation on $\bigwedge W_1^* \otimes \bigwedge W_2^*$. We omit the proof of the following Lemma which should be well known.

Lemma (1.3). *In the above situation, τ acts diagonally, i.e., if τ_1 and τ_2 denote the Chevalley operators on $\bigwedge W_1^*$ and $\bigwedge W_2^*$ respectively, then $\tau = \tau_1 \otimes \tau_2$.*

Although the $\mathfrak{sl}_2$ representation over $\bigwedge V^*$ induced by L, Λ and $H = [L, \Lambda] = \sum_k (k-n)\Pi_k$ on the standard basis matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively is not irreducible, τ reflects the eigenspaces of H, which are just the $\bigwedge^k V^*$; consequently we get isomorphisms $\tau\colon \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$. Hence, τ has the same grading behaviour as the Hodge dual. It turns out that the Chevalley operator equals the following variation of the Hodge dual.

Definition (1.4). The linear operator $\circledast\colon \bigwedge V^* \to \bigwedge V^*$ is the unique linear extension of the operators $\circledast|_{\bigwedge^k V^*} = (-1)^{\binom{k+1}{2}} *^{-1} \mathrm{I},\ k = 0, \dots, 2n$.

To prove that this linear operator indeed equals τ, we need that it splits on tensor products just as τ does.

Lemma (1.5). *Let $V^* = W_1^* \oplus W_2^*$ be an orthogonal sum. Then the variation $\circledast$ of the Hodge dual on $\bigwedge V^* = \bigwedge W_1^* \otimes \bigwedge W_2^*$ splits into the product of the $\circledast$ operators on $\bigwedge W_1^*$ and $\bigwedge W_2^*$, i.e. $\circledast(\alpha \otimes \beta) = \circledast\alpha \otimes \circledast\beta$ for all $\alpha \in \bigwedge W_1^*$, $\beta \in \bigwedge W_2^*$.*

Proof. The proof is omitted as an easy calculation on bihomogenous forms, using $*^{-1}(\alpha \otimes \beta) = (-1)^{kl} *^{-1} \alpha \otimes *^{-1}\beta$ for all $\alpha \otimes \beta \in \bigwedge^k W_1^* \otimes \bigwedge^l W_2^*$. □

Consequently, we conclude the equality of τ and $\circledast$, which turns out to be a rephrasing of the Weil identity.

Proposition (1.6). *The linear operators τ and $\circledast$ on $\bigwedge V^*$ coincide.*

Proof. By the lemmas (1.3) and (1.5) and induction on orthogonal decompositions of V^*, it suffices to consider the case of V being of dimension one. In this case, choose an orthonormal $\mathbb{R}$-basis x, y of V^*, such that $\omega = \mathrm{vol} = x \wedge y$. Then one only has to compare the values of τ and $\circledast$ on $1, x, y$ and ω which are easily seen to be equal. □

From this equality we can now derive the Weil identity (1.1) as follows.

Proof of the Weil identity (1.1). Let $\alpha \in \bigwedge^k V^*$ be a primitive k-form on V. We have

$$\exp(L)\exp(-\Lambda)\exp(L)\alpha = \tau\alpha = \circledast\alpha = (-1)^{\binom{k+1}{2}} *^{-1} \mathrm{I}\,\alpha$$

by proposition (1.6) and, using $* \exp(L) = \exp(\Lambda)*$, this easily yields

$$* \exp(L)\alpha = (-1)^{\binom{k+1}{2}} \exp(L)\exp(-\Lambda)\,\mathrm{I}\,\alpha.$$

Since α was assumed to be primitive, we get $\exp(-\Lambda)\alpha = \alpha$ and

$$\sum_{j \geqslant 0} * \frac{L^j}{j!} \alpha = * \exp(L)\alpha = (-1)^{\binom{k+1}{2}} \exp(L)\, \mathrm{I}\, \alpha = \sum_{j \geqslant 0} (-1)^{\binom{k+1}{2}} \frac{L^j}{j!}\, \mathrm{I}\, \alpha.$$

Since the Hodge dual maps k-forms to $2n-k$-forms and L is of degree 2, we conclude the theorem after comparison of the degrees and multiplication with $j!$. □

The Weil identity therefore comes from the relation between the Hodge dual and the $\mathfrak{sl}_2$ representation on $\bigwedge V^*$.

References

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